GATE 2027 EC, Section 6

Control systems

Transfer functions, block diagrams and signal flow graphs, time response, steady-state error, Routh, Nyquist, Bode and root locus, compensators, PID and state space.

36 formulas in 8 topics. Equations are numbered section.topic.formula, for example (6.1.1).

6.1 Transfer functions

Negative feedback (6.1.1)

$$T(s) = G(s)/(1 + G(s)H(s))$$

Mason's gain formula (6.1.2)

$T = (1/\Delta)\Sigma\, P_k\Delta_k$; $\Delta = 1 - \Sigma$ loops $+ \Sigma$ non-touching pairs $- \Sigma$ triples ...

Sensitivity (6.1.3)

$$S_G^{\,T} = (\partial T/T)/(\partial G/G) = 1/(1 + GH)$$

6.2 Time response

Second-order system (6.2.1)

$$T(s) = \omega_n^{\,2}/(s^2 + 2\zeta\omega_n s + \omega_n^{\,2})$$

Poles (6.2.2)

$$s = -\zeta\omega_n \pm j\omega_n\sqrt{(1-\zeta^2)};\ \omega_d = \omega_n\sqrt{(1-\zeta^2)}$$

Rise time (6.2.3)

$$t_r = (\pi - \theta)/\omega_d,\ \theta = \cos^{-1}\zeta$$

Peak time (6.2.4)

$$t_p = \pi/\omega_d$$

Peak overshoot (6.2.5)

$$M_p = e^{-\pi\zeta/\sqrt{(1-\zeta^2)}}$$

Settling time (6.2.6)

$$t_s \approx 4/(\zeta\omega_n)\ (2\%);\ 3/(\zeta\omega_n)\ (5\%)$$

First-order system (6.2.7)

Time constant τ; reaches 63.2% at τ; settles in $\approx 4\tau$

6.3 Steady-state error (unity feedback)

Error constants (6.3.1)

$$K_p = \lim_{s\to 0} G;\ K_v = \lim sG;\ K_a = \lim s^2G$$

Errors (6.3.2)

Step: $1/(1 + K_p)$; ramp: $1/K_v$; parabola: $1/K_a$

System type (6.3.3)

Type = number of open-loop poles at the origin

6.4 Stability

Routh–Hurwitz (6.4.1)

Sign changes in the first column = number of right half-plane poles

Nyquist (6.4.2)

$Z = P - N$ (N = anticlockwise encirclements of $-1 + j0$, P = open-loop RHP poles); stable iff $Z = 0$

Gain margin (6.4.3)

$GM = 1/|GH(j\omega_{pc})|$ at phase $-180°$; GM (dB) $= -20\log|GH(j\omega_{pc})|$

Phase margin (6.4.4)

$PM = 180° + \angle GH(j\omega_{gc})$ at $|GH| = 1$

PM and damping (6.4.5)

$PM \approx 100\zeta$ degrees ($\zeta < 0.7$)

6.5 Root locus

Start and end (6.5.1)

Starts at open-loop poles ($K = 0$), ends at zeros or infinity

Real axis (6.5.2)

Locus lies left of an odd number of real poles + zeros

Asymptotes (6.5.3)

Angles $(2q + 1)180°/(P - Z)$; centroid $\sigma = (\Sigma$ poles $- \Sigma$ zeros$)/(P - Z)$

Breakaway point (6.5.4)

$$dK/ds = 0$$

Angle of departure (6.5.5)

$$180° - \Sigma\angle(\text{from other poles}) + \Sigma\angle(\text{from zeros})$$

6.6 Frequency response

Bode slopes (6.6.1)

Each pole -20 dB/decade and $-90°$; each zero $+20$ dB/decade and $+90°$

Resonant peak (6.6.2)

$M_r = 1/(2\zeta\sqrt{(1-\zeta^2)})$ at $\omega_r = \omega_n\sqrt{(1-2\zeta^2)}$ ($\zeta < 0.707$)

Bandwidth (6.6.3)

$$\omega_b = \omega_n\sqrt{(1-2\zeta^2+\sqrt{(4\zeta^4-4\zeta^2+2)})}$$

6.7 Compensators and PID

Lead compensator (6.7.1)

$(1+aTs)/(1+Ts)$, $a>1$; max lead $\sin\varphi_m = (a-1)/(a+1)$ at $\omega_m = 1/(T\sqrt{a})$

Effects (6.7.2)

Lead: more phase margin, faster response; lag: smaller steady-state error

PID controller (6.7.3)

$$C(s) = K_p + K_i/s + K_d\, s$$

PID roles (6.7.4)

P speeds up; I removes steady-state error; D adds damping

6.8 State space

Model (6.8.1)

$$\dot{x} = Ax + Bu;\ y = Cx + Du$$

Transfer function (6.8.2)

$T(s) = C(sI-A)^{-1}B + D$; poles = eigenvalues of A

State transition matrix (6.8.3)

$$\Phi(t) = e^{At} = L^{-1}\{(sI-A)^{-1}\}$$

Solution (6.8.4)

$$x(t) = \Phi(t)x(0) + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau$$

Controllability (6.8.5)

$$\text{rank}[B\ AB\ \ldots\ A^{n-1}B] = n$$

Observability (6.8.6)

$$\text{rank}[C;\ CA;\ \ldots;\ CA^{n-1}] = n$$

My notes